[3] This way of defining measure in function space was discussed by Kolmogoroff, *Ergebnisse Mathematik*, **2,** No. 3, Grundbegriffe der Wahrscheinlichkeitsrechnung, § 4.

[4] It is sufficient that if E is any set in Ω determined by conditions of the form (1′), and if E is transformed into E_t by T_t, the measure of $E \cdot E_t$ should be continuous in t at $t = 0$.

[5] For a simple proof of the ergodic theorem, following the lines of the first proof, given by Birkhoff, cf. A. Khintchine, *Mathemat. Ann.*, **107,** 485–488 (1933).

[6] This situation was discussed by Khintchine, *Zeit. Angewandte Mathemat. Mechanik*, **13,** 101–103 (1933), who treated the particular case of chance variables taking on only the values 1 or 0. The general case was discussed by E. Hopf, *Journal of Mathematics and Physics*, M. I. T., **13,** 51–102 (1934), who obtained (3′) but not Theorem 2.

[7] Kolmogoroff, loc. cit.,[3] p. 59, announced this result in the special case of independent chance variables, and announced also Theorem 2, under the assumption that the probability is 1 that the upper limit in (4) is 0.

[8] Loc. cit.,[5] p. 488.

[9] If $f(x)$ is defined for $-\infty < x < \infty$ except possibly for a set of points of Lebesgue measure 0, is Lebesgue measurable, not negative, and integrable over $-\infty < x < \infty$, $\int_{-\infty}^{\infty} f(x)dx = 1$, $f(x)$ will be called a probability density.

[10] This method was discussed (unrigorously) by Fisher in the *Phil. Trans. Roy. Soc. London*, Series A, **222** (1921). The treatment of H. Hotelling, *Trans. Amer. Math. Soc.*, **32,** 847–859 (1930), holds only in certain special cases.

* NATIONAL RESEARCH FELLOW.

REMARKS ON THE POSSIBLE FAILURE OF ENERGY CONSERVATION

BY RICHARD C. TOLMAN

NORMAN BRIDGE LABORATORY OF PHYSICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated April 17, 1934

1. *Possibility of Failure in the Case of Elementary Processes.*—The possible failure of the principle of the conservation of energy, in the case of the continuous β-ray spectrum accompanying radioactive decompositions, and perhaps also in the case of processes occurring in the interior of stars, has several times been suggested by Bohr.[1] From a theoretical point of view such a failure might be due to a breakdown in the applicability of ordinary mechanical notions under circumstances where the electron would have to be regarded as localizable within regions small compared with its classical dimensions.

In the case of the stars there is at present no definite observational evidence which would lead us to abandon the principle of the conservation of energy, beyond the removal of limitations on our attempts to explain the continued luminosity of those objects and to account, in general, for the existence of a supply of available energy in the universe.

In the case of the continuous β-ray spectrum, there are, however, two important observations which might make an abandonment of the principle of conservation seem attractive. In the first place, the β-rays, emitted in natural radioactive processes when the atomic number increases by one, are found to exhibit a continuous spectrum of velocities with an observed spread in kinetic energy which may amount to some million electron-volts.[2] In the second place, the total calorimetrically measured energy made available by such processes appears to agree with the average energy calculated from the spectrum,[3] so that it is not possible to assume that all the electrons are really emitted with the same energy and then lose varying amounts through collision before the determination of their velocities. These observations indicate that different atoms of the parent element can decompose to give atoms of the next element in the radioactive series by the emission of electrons having widely different energies.

To retain the principle of the conservation of energy in the light of these findings, we might of course assume that different atoms of a given chemical isotope have nuclei which are not really exactly alike, so that different amounts of energy actually are available. In opposition to such an assumption, nevertheless, we have such facts as the sharpness of γ-ray levels, and the constancy in rate of decay both for α- and β-particle disintegrations which give some evidence for the identical nature of all the nuclei of a given isotope.

To retain the principle of conservation, we might also assume as an alternative explanation that the emission of electrons is not the sole process accompanying a β-ray decomposition, but in addition that some very penetrating radiation is simultaneously emitted which carries off the balance of energy left by the electrons, and then escapes through the walls of the container without being calorimetrically detected. For this purpose neutrons of very small mass have been postulated.[4] At present, nevertheless, there are no additional facts to support such an hypothesis.

These observational facts as to natural β-ray disintegrations hence make the possibility of failures in energy conservation worthy of consideration. Furthermore, the recent discovery of artificially produced radioactive substances[5] has been found to provide cases in which positive electrons are emitted also with a continuous range of energies, as shown particularly by the work of Lauritsen and Crane[6] and Neddermeyer and Anderson.[7] Moreover, since the previous history of the artificially produced radioactive substances is experimentally controllable, these may prove superior to natural radioactive substances for further empirical tests of the failure or validity of the principle of conservation.

2. *Possibility of Failure from a Statistical Point of View.*—The foregoing familiar remarks as to the continuous β-ray spectrum have indicated the possibility that similar atoms of a given parent substance might decompose

into the same end products but with the liberation of varying amounts of energy. This result would violate the principle of the conservation of energy for individual elementary processes. It would not, however, necessarily violate the conservation of energy from a statistical point of view, since the average energy liberated in the decomposition of many such atoms might still be equal to the difference between the intrinsic energies of the initial and final substances, as measured for example by their difference in mass.

Nevertheless, as pointed out to me by Professor Bohr in conversation, the ejection of electrons having a definite wide range of energies from nuclei all of which are alike would also involve a statistical failure in energy conservation, if we make the additional assumption of a finite probability for electrons of different energies to reënter the nucleus and rebuild the parent substance. Under these circumstances we could then obtain an actual net increase in energy by allowing a decomposition to take place and then rebuilding the original nuclei using electrons having lower energies than the average of those that were spontaneously emitted.

As a specific example of this possibility, which might even present some interest for stellar theory, let us consider a system consisting of an enclosed gas containing atoms and electrons which can enter into the *reversible nuclear* reaction

$$N \rightleftarrows N^+ + E^-, \tag{1}$$

and let us allow the system to come to a steady state such that the number of nuclei N which break down in unit time to give nuclei N^+ and emitted electrons E^- is balanced by the reverse reaction, and such that any accompanying net production or disappearance of energy is balanced by the existing rate of interchange with the surroundings.

Assuming conditions such that the distribution of kinetic energy among the free electrons in the system corresponds closely to the Maxwell distribution at the temperature T of the gas, we can then write as an approximate expression for the rate of change in the concentration of nuclei N

$$-\frac{d(N)}{dt} = k_1(N) - (N^+)(E^-)\frac{2}{(\pi k^3T^3)^{1/2}}\int_0^\infty \phi(\epsilon)e^{-\epsilon/kT}\epsilon^{1/2}d\epsilon = 0, \tag{2}$$

where (N), (N^+) and (E^-) are the concentrations of the particles indicated, k_1 is the constant for the unimolecular rate of decomposition of the nuclei N, and $\phi(\epsilon)$ is the chance per unit time and per unit volume for an electron of kinetic energy ϵ to collide with a nucleus N^+ and rebuild the parent substance. This rate has been equated to zero in agreement with the steady state chosen for consideration.

Furthermore, for the net rate of energy production we can evidently write

$$\frac{dU}{dt} = k_1(N)\bar{\epsilon} - (N^+)(E^-)\frac{2}{(\pi k^3T^3)^{1/2}}\int_0^\infty \phi(\epsilon)e^{-\epsilon/kT}\epsilon^{3/2}d\epsilon, \tag{3}$$

where $\bar{\epsilon}$ is the constant average energy of the ejected electrons. In agreement with our choice of a steady state for discussion, this will be the rate of energy transfer from the system to its surroundings.

In the absence of knowledge as to the functional form of $\phi(\epsilon)$, we cannot definitely evaluate this rate of energy production and exchange with the surroundings. Nevertheless, by combining (2) and (3) we see that the condition for zero rate of energy production would be

$$\bar{\epsilon}\int_0^\infty \phi(\epsilon)e^{-\epsilon/kT}\epsilon^{1/2}d\epsilon = \int_0^\infty \phi(\epsilon)e^{-\epsilon/kT}\epsilon^{3/2}d\epsilon, \qquad (4)$$

and with $\bar{\epsilon}$ a constant independent of T this could be true only under special circumstances. With $\phi(\epsilon)$ arbitrary the equality could be satisfied at some one particular temperature, but to secure zero rate of energy production under all circumstances it is evident that $\phi(\epsilon)$ would have to be zero for all value of ϵ except for the value $\bar{\epsilon}$. This possibility of still rescuing the principle of the conservation of energy from a statistical point of view, by assuming that nuclei have a finite probability of forming only when the total energy available is equal to the average energy which will again be made available by their later decomposition, is an important one to consider and to keep in mind in experiments on artificially produced radioactive substances. Nevertheless, prior to empirical test, the assumption appears artificial.

3. *Conclusion.*—In conclusion three points may be mentioned in connection with the foregoing discussion.

In the first place, it is of course evident that the failure or validity of our older ideas as to the conservation of energy is a matter for experimental test. This note deals with conceptual possibilities and no opinion is intended as to the ultimate empirical decision.

In the second place, even if the experimental outcome should indicate that energy in its familiar forms can be created and destroyed by such processes as discussed above, it should be noted that the principle of conservation might perhaps still be preserved by the device of adding to the expression for energy a new term purposely so chosen as to maintain conservation. Nevertheless, such a rescue of the principle of energy conservation might be purely formal and of little real convenience.[8] The experience of general relativity in this connection is instructive, where the conservation of energy and momentum can be preserved in all coördinates by Einstein's introduction of the pseudo-tensor density of potential energy and momentum t_μ^ν, whose components nevertheless are dependent on the actual distribution of matter and radiation in such a complicated way that we do not often make use of this possibility.

As a final remark, it is evident that the creation and destruction of energy in its ordinary forms by microscopic processes such as those discussed above would also involve modifications in the equations of macroscopic relativistic mechanics, since these would allow no creation or destruction of energy from the point of view of a local observer. These modifications might prove of interest for the problems of relativistic cosmology. The possible nature of the changes which could be introduced will be discussed in a following note.

[1] Bohr, *Jour. Chem. Soc.*, 349 (1932); also *Atomic Stability and Conservation Laws*, Reale Accademia d'Italia, Rome (1932).

[2] See the collection of data given by Rutherford, Chadwick and Ellis, *Radiations from Radioactive Substances*, Cambridge (1930).

[3] Ellis and Wooster, *Proc. Roy. Soc.*, **A117**, 109 (1927); Meitner and Orthmann, *Zeit. Phys.*, **60**, 143 (1930).

[4] Pauli, Paper before the Am. Phys. Soc., Pasadena, June 16, 1931; Fermi, *La Ricerca Scientifica*, Anno IV, **2**, No. 12 (1933); *Zeit. Phys.*, **88**, 161(1934).

[5] Curie and Joliot, *Compt. Rend.*, **198**, 254 (1934).

[6] Lauritsen and Crane, *Phys. Rev.*, **45**, 497 (1934).

[7] Neddermeyer and Anderson, *Ibid.*, **45**, 498 (1934).

[8] See the discussion of Poincaré, *Science and Hypothesis*, London (1905).

THE DIRAC EQUATION IN PROJECTIVE RELATIVITY

BY A. H. TAUB, O. VEBLEN AND J. VON NEUMANN

PRINCETON UNIVERSITY AND THE INSTITUTE FOR ADVANCED STUDY

Communicated May 9, 1934

1. In this note we discuss the extension of the Dirac equation to general relativity. In order to have equations which are automatically invariant with respect to gauge as well as coördinate and spin transformations, we use the method of projective relativity. This theory is physically equivalent to the original general relativity theory of an Einstein gravitational and a Maxwell electromagnetic field.

We are led to a class of equations of the Dirac type. One of these reduces exactly to the Dirac equation of a charged particle in special relativity. This equation is identical with the one given by Schrödinger[1] and therefore equivalent to the one given by Fock.[2] The others contain extra terms which correspond to physical situations in which the field of a dipole is superposed on the field of the charge. Such extra terms, with the charge 0, have been proposed by Pauli in order to explain the properties of the neutron.[3] The class thus contains equations first proposed by Schouten and Van Dantzig,[4] in which an extra term appeared. One of these equations coincides with an equation proposed by Pauli,[5] and may be considered as the simplest one in the projective notation, while the one without the extra term is simpler in the affine notation (equation (3.3)).